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T-duality in 2D integrable models

J F Gomes, G M Sotkov and A H Zimerman

Instituto de Física Teórica—IFT/UNESP, Rua Pamplona 145, 01405-900, São Paulo, SP, Brazil

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Abstract
The non-conformal analogue of Abelian T-duality transformations relating pairs of axial and vector integrable models from the non-Abelian affine Toda family is constructed and studied in detail.

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1. Introduction

Abelian T-duality in $U(1)^{\otimes s}$ invariant 2D conformal field theories (CFTs) and in string theory represents a set of specific canonical transformations that relate pairs of equivalent models sharing the same spectrum, but with different σ-model-like Lagrangians [1, 2]. The axial and vector gauged G/H-WZW models provide a vast variety of examples of such pairs of T-dual models [3, 4]. On the other hand, the integrable perturbations of these G/H-WZW models have been identified with the family of the so-called non-Abelian affine Toda theories [5–7]. An important feature of these integrable models (IMs) is their $U(1)^{\otimes k}$, $k \leqslant s$, global symmetry and the fact that they admit both topological and/or non-topological soliton solutions carrying $U(1)^{\otimes k}$ charges as well [7, 8]. Hence, an interesting problem to be addressed is about the T-duality of pairs of axial and vector IMs within this family. More precisely, whether the perturbation breaks a part (or all) of the isometries (i.e. $U(1)^{\otimes s_{\mathrm{CFT}}}$ to $U(1)^{\otimes s_{\mathrm{Im}}}$, $s_{\mathrm{Im}} \leqslant s_{\mathrm{CFT}}$) and whether certain *non-conformal* analogues of the Abelian T-duality transformations take place. The simplest example of a pair of T-dual IMs with only one isometry (i.e. $s_{\mathrm{Im}} = 1$) has been studied in detail in our recent paper [7, 9]. As one expects, the mass spectrum of the solitons is indeed invariant under the corresponding non-critical T-duality, but the $U(1)$-charges of the solitons of the axial model are mapped into the topological charges of the solitons of the vector IM and vice versa. An interesting example of T-self-dual IMs is given by the complex sine-Gordon [6] and the Fateev IMs [10].

The present paper is devoted to the investigation of T-duality properties of the family of IMs representing relativistic IM belonging to the same hierarchy as the Fordy–Kulish (multi-component) nonlinear Schrödinger model (NLS) [11, 12]. They can be considered as a specific Hamiltonian reduction of the $A_n^{(1)}$-homogeneous sine-Gordon models [13]. Their main property is the large global symmetry group $SL(N) \otimes U(1)$, i.e. they admit N-isometries, as in the $SL(N+1)/SL(N)\otimes U(1)$-WZW model. As a consequence the T-duality transformations relating the corresponding axial and vector IMs of this family are indeed more involved.

The paper is organized as follows. Section 2 contains a brief summary of the general formalism for the construction of the effective action of a large class of NA-affine Toda theories. In section 3, we apply these methods for the derivation of the Lagrangians of axial and vector IMs of reduced homogeneous SG-type. Section 4 is devoted to the symmetries of such models while in section 5 we explicitly construct the corresponding T-duality transformations.

2. NA affine Toda models as gauged two-loop WZW models

The basic ingredient in constructing massive Toda models is the decomposition of an affine Lie algebra $\mathcal{G}$ in terms of graded subspaces defined according to a grading operator Q,

$$[Q,\mathcal{G}_l]=l\mathcal{G}_l \qquad \mathcal{G}=\oplus\mathcal{G}_l \qquad [\mathcal{G}_l,\mathcal{G}_k]\subset\mathcal{G}_{l+k}, \qquad l,k=0,\pm1,\ldots. \tag{2.1}$$

In particular, the zero grade subspace $\mathcal{G}_0$ plays an important role since it is parametrized by the Toda fields. The grading operator Q induces the notion of negative ($\mathcal{G}_<$) and positive ($\mathcal{G}_>$) grade subalgebras and henceforth the decomposition of a group element in the Gauss form,

$$g=NBM \tag{2.2}$$

where $N=\exp(\mathcal{G}_<)$, $B=\exp(\mathcal{G}_0)$ and $M=\exp(\mathcal{G}_>)$.

The action of the corresponding affine Toda models can be derived from the gauged two-loop[1] Wess–Zumino–Witten (WZW) action [15, 7],

$$S_{G/H}(g,A,\bar{A})=S_{\mathrm{WZW}}(g)-\frac{k}{2\pi}\int \mathrm{d}^2x\,\mathrm{Tr}(A(\bar{\partial}gg^{-1}-\epsilon_+)+\bar{A}(g^{-1}\partial g-\epsilon_-)+Ag\bar{A}g^{-1}) \tag{2.3}$$

where $A=A_-\in\mathcal{G}_<$, $\bar{A}=\bar{A}_+\in\mathcal{G}_>$ and $\epsilon_\pm$ are constant elements of grade ±1. The action (2.3) is invariant under

$$g'=\alpha_-g\alpha_+ \qquad A'=\alpha_-A\alpha_-^{-1}+\alpha_-\partial\alpha_-^{-1} \qquad \bar{A}'=\alpha_+^{-1}\bar{A}\alpha_++\bar{\partial}\alpha_+^{-1}\alpha_+ \tag{2.4}$$

where $\alpha_-\in G_<,\alpha_+\in G_>$. It therefore follows that $S_{G/H}(g,A,\bar{A})=S_{G/H}(B,A',\bar{A}')$.

Integrating over the auxiliary fields A and $\bar{A}$ in the partition function

$$Z=\int DAD\bar{A}DB\,\mathrm{e}^{-S} \tag{2.5}$$

we find the effective action for an integrable model defined on the group G_0,

$$S_{\mathrm{eff}}(B)=S_{\mathrm{WZW}}(B)-\frac{k}{2\pi}\int\mathrm{Tr}(\epsilon_+B\epsilon_-B^{-1})\,\mathrm{d}^2x. \tag{2.6}$$

The corresponding equations of motion have the following compact form [16]:

$$\bar{\partial}(B^{-1}\partial B)+[\epsilon_-,B^{-1}\epsilon_+B]=0 \qquad \partial(\bar{\partial}BB^{-1})-[\epsilon_+,B\epsilon_-B^{-1}]=0. \tag{2.7}$$

It is straightforward to derive from equations (2.7) the chiral conserved currents associated with the subalgebra $\mathcal{G}_0^0\subset\mathcal{G}_0$ defined as $\mathcal{G}_0^0=\{X\in\mathcal{G}_0,\text{ such that }[X,\epsilon_\pm]=0\}$, i.e.

$$J_X=\mathrm{Tr}(XB^{-1}\partial B) \qquad \bar{J}_X=\mathrm{Tr}(X\bar{\partial}BB^{-1}) \qquad \bar{\partial}J_X=\partial\bar{J}_X=0. \tag{2.8}$$

The conservation of such currents is a consequence of the invariance of the action (2.6) under the $G_0^0\otimes G_0^0$ chiral transformation,

$$B'=\bar{\Omega}(\bar{z})B\Omega(z) \tag{2.9}$$

where $\bar{\Omega}(\bar{z}),\Omega(z)\in G_0^0$.

[1] The $\hat{G}$-WZW model in the case where $\hat{\mathcal{G}}$ is an affine Kac–Moody algebra is called the two-loop WZW model [14].

The fact that the currents J_X and $\bar{J}_X$ in (2.8) are chiral, allows further reduction of the IM (2.6) by imposing a set of subsidiary constraints,

$$J_X = \mathrm{Tr}(XB^{-1}\partial B) = 0 \qquad \bar{J}_X = \mathrm{Tr}(X\bar{\partial}BB^{-1}) = 0 \quad X \in \mathcal{G}_0^0 \tag{2.10}$$

which reduces the model defined on the group G_0 to the one on coset G_0/G_0^0. Such constraints are incorporated into the action by repeating the gauged WZW action argument for the subgroup G_0. For a general non-Abelian $\mathcal{G}_0^0$ we define a second grading structure Q' which decomposes $\mathcal{G}_0^0$ into positive, zero and negative graded subspaces, i.e. $\mathcal{G}_0^0 = \mathcal{G}_0^{0,<} \oplus \mathcal{G}_0^{0,0} \oplus \mathcal{G}_0^{0,>}$. Following the same principle as in [15, 7, 8] we seek for an action invariant under

$$B'' = \gamma_0(\bar{z}, z)\gamma_-(\bar{z}, z)B\gamma_+(\bar{z}, z)\gamma_0'(\bar{z}, z) \qquad \gamma_0, \gamma_0' \in G_0^{0,0} \quad \gamma_- \in G_0^{0,<} \quad \gamma_+ \in G_0^{0,>} \tag{2.11}$$

and choose $\gamma_0(\bar{z}, z), \gamma_0'(\bar{z}, z), \gamma_-(\bar{z}, z), \gamma_+(\bar{z}, z) \in G_0^0$ such that $B'' = \gamma_0\gamma_-B\gamma_+\gamma_0' = g_0^f \in G_0/G_0^0$ since B can also be decomposed into the Gauss form according to the second grading structure Q'. Denote $\Gamma_- = \gamma_0\gamma_-$ and $\Gamma_+ = \gamma_+\gamma_0'$. Then the action

$$\begin{aligned} S(B, A^{(0)}, \bar{A}^{(0)}) = S_{\mathrm{WZW}}(B) - \frac{k}{2\pi}\int \mathrm{Tr}(\epsilon_+B\epsilon_-B^{-1})\,\mathrm{d}^2x - \frac{k}{2\pi}\int \mathrm{Tr}\big(\eta A^{(0)}\bar{\partial}BB^{-1} \\ + \bar{A}^{(0)}B^{-1}\partial B + \eta A^{(0)}B\bar{A}^{(0)}B^{-1} + A_0^{(0)}\bar{A}_0^{(0)}\big)\,\mathrm{d}^2x \end{aligned} \tag{2.12}$$

(with $\eta = +1, -1$ correspond to $\gamma_0' = \gamma_0$ for axial or $\gamma_0' = \gamma_0^{-1}$ for vector gaugings[2] respectively, $A^{(0)} = A_0^{(0)} + A_-^{(0)}$ and $\bar{A}^{(0)} = \bar{A}_0^{(0)} + \bar{A}_+^{(0)}$), is invariant under $\Gamma_\pm$ transformations

$$\begin{aligned} &B' = \Gamma_-B\Gamma_+ \qquad A_0'^{0} = A_0^{(0)} - \eta\gamma_0^{-1}\partial\gamma_0 \qquad \bar{A}_0'^{0} = \bar{A}_0^{(0)} - \gamma_0^{-1}\bar{\partial}\gamma_0 \\ &A'^{(0)} = \Gamma_-A^{(0)}\Gamma_-^{-1} - \eta\partial\Gamma_-\Gamma_-^{-1} \qquad \bar{A}'^{(0)} = \Gamma_+^{-1}\bar{A}^{(0)}\Gamma_+ - \Gamma_+^{-1}\bar{\partial}\Gamma_+ \end{aligned} \tag{2.13}$$

where $A_0^{(0)}, \bar{A}_0^{(0)} \in \mathcal{G}_0^{0,0}$, $A_-^{(0)} \in \mathcal{G}_0^{0,<}$, $\bar{A}_+^{(0)} \in \mathcal{G}_0^{0,>}$. Hence we have

$$S(B, A^{(0)}, \bar{A}^{(0)}) = S\big(g_0^f, A'^{(0)}, \bar{A}'^{(0)}\big). \tag{2.14}$$

The general construction above provides a systematic classification of relativistic integrable models in terms of its algebraic structure, i.e. $\{\mathcal{G}, Q, \epsilon_\pm, \mathcal{G}_0^0\}$. For example, within the affine $\mathcal{G} = \hat{SL}(N+1)$ algebra we have the following families of integrable models:

(1) $\mathcal{G}_0^0 = \emptyset$ characterizes the choices of

$$Q = (N+1)d + \sum_{l=1}^{N}\lambda_l\cdot H \qquad \mathcal{G}_0 = U(1)^N = \{h_1, \ldots, h_N\}$$

$$\epsilon_\pm = \mu\left(\sum_{l=1}^{N}E_{\pm\alpha_l}^{(0)} + E_{\mp(\alpha_1+\cdots+\alpha_N)}^{(\pm1)}\right)$$

which gives rise to the well-known Abelian affine Toda model (see for instance [17, 16]).

(2)

(a) $\mathcal{G}_0^0 = U(1) = \{\lambda_1\cdot H\}$

$$Q = Nd + \sum_{l=2}^{N}\lambda_l\cdot H \qquad \mathcal{G}_0 = SL(2)\otimes U(1)^{N-1} = \big\{E_{\pm\alpha_1}, h_1, \ldots, h_N\big\}$$

$$\epsilon_\pm = \mu\left(\sum_{l=2}^{N}E_{\pm\alpha_l}^{(0)} + E_{\mp(\alpha_2+\cdots+\alpha_N)}^{(\pm1)}\right)$$

corresponds to the simplest non-Abelian affine Toda model of dyonic type, admitting electrically charged topological solitons (see for instance [7, 15]).

[2] Note that for non-Abelian $\mathcal{G}_0^0$ the invariance of the vector action in (2.12) is a consequence of the Borel structure of the subgroup elements $\Gamma_\pm$, i.e. we consider the left–right coset $\Gamma_-\backslash G/\Gamma_+$.

(b) $\mathcal{G}_0^0 = U(1) \otimes U(1) = \{\lambda_1 \cdot H, \lambda_N \cdot H\}$

$$Q = (n-1)d + \sum_{l=2}^{N-1} \lambda_l \cdot H \qquad \epsilon_\pm = \mu \left(\sum_{l=2}^{N-1} E^{(0)}_{\pm\alpha_l} + E^{(\pm 1)}_{\mp(\alpha_2+\cdots+\alpha_{N-1})} \right)$$

$$\mathcal{G}_0 = SL(2) \otimes SL(2) \otimes U(1)^{N-2} = \left\{ E_{\pm\alpha_1}, E_{\pm\alpha_N}, h_1, \ldots, h_N \right\}$$

is of the same class of $U(1)^{\otimes k}$ dyonic type IMs, but now yielding multicharged solitons ([8]).

(3) $\mathcal{G}_0^0 = SL(2) \otimes U(1) = \left\{ E_{\pm\alpha_1}, \lambda_1 \cdot H, \lambda_2 \cdot H \right\}$

$$Q = (N-1)d + \sum_{l=3}^{N} \lambda_l \cdot H \qquad \epsilon_\pm = \mu \left(\sum_{l=3}^{N} E^{(0)}_{\pm\alpha_l} + E^{(\pm 1)}_{\mp(\alpha_3+\cdots+\alpha_N)} \right)$$

$$\mathcal{G}_0 = SL(3) \otimes U(1)^{N-2} = \left\{ E_{\pm\alpha_1}, E_{\pm\alpha_2}, E_{\pm(\alpha_1+\alpha_2)}, h_1, \ldots, h_N \right\}$$

and $Q' = \lambda_1 \cdot H$, such that $\mathcal{G}_0^{0,<} = \left\{ E_{-\alpha_1} \right\}$, $\mathcal{G}_0^{0,>} = \left\{ E_{\alpha_1} \right\}$, $\mathcal{G}_0^{0,0} = \{\lambda_1 \cdot H, \lambda_2 \cdot H\}$ leads to dyonic models with non-Abelian global symmetries (see section 6 of [8]).

The classical integrability of all these models follows from their zero curvature (Lax) representation:

$$\partial \bar{\mathcal{A}} - \bar{\partial} \mathcal{A} - [\mathcal{A}, \bar{\mathcal{A}}] = x0 \qquad \mathcal{A}, \bar{\mathcal{A}} \in \oplus_{i=0,\pm 1} \mathcal{G}_i \tag{2.15}$$

with

$$\mathcal{A} = -B\epsilon_- B^{-1} \qquad \bar{\mathcal{A}} = \epsilon_+ + \bar{\partial} B B^{-1} \tag{2.16}$$

where the constraints (2.10) are imposed. It can be easily verified that substituting (2.16) into (2.15) taking into account (2.10), one reproduces the equations of motion (2.7). Then the existence of an infinite set (of commuting) conserved charges $P_m, m = 0, 1, \ldots$ is a simple consequence of equation (2.15), namely,

$$P_m(t) = \mathrm{Tr}(T(t))^m \qquad \partial_t P_m = 0 \qquad T(t) = \lim_{L\to\infty} \mathcal{P} \exp \int_{-L}^{L} \mathcal{A}_x(t,x)\, \mathrm{d}x.$$

Hence the above-described procedure for derivation of the Abelian and NA affine Toda models as gauged G/H two-loop WZW models leads to integrable models by construction.

3. Homogeneous gradation and the Lund–Regge type models

An interesting class of integrable models, that generalizes the Lund–Regge model [18], can be constructed from the affine Kac–Moody algebra $\hat{\mathcal{G}} = \hat{SL}(N+1)$ endowed with homogeneous gradation $Q = d$ and the specific choice of $\epsilon_\pm = \mu \lambda_N \cdot H^{(\pm 1)}$, where λ_N is the Nth fundamental weight of $SL(N+1)$. The zero grade subalgebra $\mathcal{G}_0$ corresponds to the finite-dimensional Lie algebra $\mathcal{G}_0 = SL(N+1)$ and $\mathcal{G}_0^0 = SL(N) \otimes U(1)$. Let us parametrize the auxiliary gauge fields as follows:

$$\begin{aligned} A_0^{(0)} &= \sum_{i=1}^{N} a_i (\lambda_i - \lambda_{i-1}) \cdot H^{(0)} \qquad \bar{A}_0^{(0)} = \sum_{i=1}^{N} \bar{a}_i (\lambda_i - \lambda_{i-1}) \cdot H^{(0)} \qquad \lambda_0 = 0 \\ A_-^{(0)} &= \sum_{j=1}^{N-1} \sum_{i=j}^{N-1} a_{i+1,j} E^{(0)}_{-(\alpha_j+\cdots+\alpha_i)} \qquad \bar{A}_+^{(0)} = \sum_{j=1}^{N-1} \sum_{i=j}^{N-1} \bar{a}_{j,i+1} E^{(0)}_{\alpha_j+\cdots+\alpha_i} \end{aligned} \tag{3.17}$$

where $a_{ij}(x,t), a_i(x,t), \bar{a}_{ij}(x,t), \bar{a}_i(x,t)$ are arbitrary functions of spacetime variables. We next consider two different gauge fixings of $\mathcal{G}_0^0$, the vector and the axial, in order to derive the effective Lagrangians for the pair of T-dual IMs.

3.1. Axial gauging

According to the axial gauging (2.11), $\eta = 1$, $\gamma_0' = \gamma_0$, the factor group element $g_0^f \in G_0 / G_0^0$ is parametrized as follows:

$$g_0^f = g_{0,ax}^f = nm \qquad n = \exp\left(\sum_{i=1}^{N} \chi_i E_{-(\alpha_i+\cdots+\alpha_N)}\right) \qquad m = \exp\left(\sum_{i=1}^{N} \psi_i E_{\alpha_i+\cdots+\alpha_N}\right). \tag{3.18}$$

After a tedious but straightforward calculation we find

$$\begin{aligned}\mathrm{Tr}\big(A_0^{(0)}\bar{A}_0^{(0)} + A^{(0)} g_0^f \bar{A}^{(0)} g_0^{f\,-1} + A^{(0)}\bar{\partial} g_0^f g_0^{f\,-1} + \bar{A}^{(0)} g_0^{f\,-1}\partial g_0^f\big) \\ = \bar{a}_i M_{ij} a_j + \bar{a}_i N_i + \bar{N}_i a_i + \sum_{j=1}^{N-1}\sum_{i=j}^{N-1}\sum_{k=j}^{N-1} \bar{a}_{j,i+1} a_{k+1,j}(\delta_{i,k} + \psi_{i+1}\chi_{k+1}) \\ - \sum_{j=1}^{N-1}\sum_{i=j}^{N-1} \bar{a}_{j,i+1}\psi_{i+1}\partial\chi_j - \sum_{j=1}^{N-1}\sum_{i=j}^{N-1} a_{i+1,j}\chi_{i+1}\bar{\partial}\psi_j\end{aligned} \tag{3.19}$$

where we have introduced M_{ij} and $N_j, \bar{N}_j$ as

$$\begin{aligned}&M_{i,j} = 2(\lambda_i - \lambda_{i-1})\cdot(\lambda_j - \lambda_{j-1}) + \psi_i\chi_i\delta_{i,j} \qquad i, j = 1, \ldots, N \quad \lambda_0 = 0 \\ &N_j = \left(\sum_{i=j}^{N-1} a_{i+1,j}\chi_{i+1} - \partial\chi_j\right)\psi_j \qquad \bar{N}_j = \left(\sum_{i=j}^{N-1} \bar{a}_{j,i+1}\psi_{i+1} - \bar{\partial}\psi_j\right)\chi_j.\end{aligned} \tag{3.20}$$

In order to derive the effective Lagrangian of the axial model we have to integrate the auxiliary fields $a_1, \bar{a}_i, a_{j,i+1}$ and $\bar{a}_{i+1,j}$. We shall consider the particular case $N = 2$, i.e. $\mathcal{G} = \hat{SL}(3)$, where the Gaussian matrix integration is quite simple. Then, in the parametrization (3.18)

$$\begin{aligned}B &= \mathrm{e}^{\tilde{\chi}_1 E_{-\alpha_1}}\, \mathrm{e}^{\tilde{\chi}_2 E_{-\alpha_2} + \tilde{\chi}_3 E_{-\alpha_1-\alpha_2}}\, \mathrm{e}^{\phi_1 h_1 + \phi_2 h_2}\, \mathrm{e}^{\tilde{\psi}_2 E_{\alpha_2} + \tilde{\psi}_3 E_{\alpha_1+\alpha_2}}\, \mathrm{e}^{\tilde{\psi}_1 E_{\alpha_1}} \\ &= \mathrm{e}^{\tilde{\chi}_1 E_{-\alpha_1}}\, \mathrm{e}^{\frac{1}{2}(\lambda_1\cdot HR_1 + \lambda_2\cdot HR_2)}\big(g_{0,ax}^f\big)\, \mathrm{e}^{\frac{1}{2}(\lambda_1\cdot HR_1 + \lambda_2\cdot HR_2)}\, \mathrm{e}^{\tilde{\psi}_1 E_{\alpha_1}} \\ g_{0,ax}^f &= \mathrm{e}^{\chi_1 E_{-\alpha_1-\alpha_2} + \chi_2 E_{-\alpha_2}}\, \mathrm{e}^{\psi_1 E_{\alpha_1+\alpha_2} + \psi_2 E_{\alpha_2}} \\ \phi_1 h_1 &+ \phi_2 h_2 = \lambda_1\cdot HR_1 + \lambda_2\cdot HR_2\end{aligned} \tag{3.21}$$

we have $M_{ij}, N_j\bar{N}_j, i, j = 1, 2$ in the form

$$M = \begin{pmatrix} \frac{4}{3} + \psi_1\chi_1 & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} + \psi_2\chi_2 \end{pmatrix} \tag{3.22}$$

and

$$\bar{N} = (-(\bar{\partial}\psi_1 - \bar{a}_{1,2}\psi_2)\chi_1, -(\chi_2\bar{\partial}\psi_2)) \qquad N = \begin{pmatrix} -(\partial\chi_1 - a_{2,1}\chi_2)\psi_1 \\ -(\psi_1\partial\chi_1) \end{pmatrix}. \tag{3.23}$$

Integrating first over the a_i and $\bar{a}_i$ and next on the a_{12}, $\bar{a}_{21}$ we derive the effective action of the $SL(3)$ axial model

$$\begin{aligned}S_{\mathrm{ax}} = -\frac{k}{2\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\bigg(\frac{1}{\Delta}\bigg(\bar{\partial}\psi_2\partial\chi_2(1 + \psi_1\chi_1 + \psi_2\chi_2) + \bar{\partial}\psi_1\partial\chi_1(1 + \psi_2\chi_2) \\ - \frac{1}{2}(\psi_2\chi_1\bar{\partial}\psi_1\partial\chi_2 + \chi_2\psi_1\bar{\partial}\psi_2\partial\chi_1)\bigg) - V\bigg)\end{aligned} \tag{3.24}$$

where $V = \mu^2\big(\frac{2}{3} + \psi_1\chi_1 + \psi_2\chi_2\big)$ and $\Delta = (1 + \psi_2\chi_2)^2 + \psi_1\chi_1\big(1 + \frac{3}{4}\psi_2\chi_2\big)$.

3.2. Vector gauging

For the explicit $SL(3)$ case, the zero grade group element B is written according to the vector gauging $\left(\eta = -1, \gamma_0' = \gamma_0^{-1}\right)$ as

$$\begin{aligned} B &= \mathrm{e}^{\tilde{\chi}_1 E_{-\alpha_1}} \, \mathrm{e}^{\tilde{\chi}_2 E_{-\alpha_2} + \tilde{\chi}_3 E_{-\alpha_1-\alpha_2}} \, \mathrm{e}^{\phi_1 h_1 + \phi_2 h_2} \, \mathrm{e}^{\tilde{\psi}_2 E_{\alpha_2} + \tilde{\psi}_3 E_{\alpha_1+\alpha_2}} \, \mathrm{e}^{\tilde{\psi}_1 E_{\alpha_1}} \\ &= \mathrm{e}^{\tilde{\chi}_1 E_{-\alpha_1}} \, \mathrm{e}^{\frac{1}{2}(\lambda_1 \cdot H u_1 + \lambda_2 \cdot H u_2)} \left(g^f_{0,\mathrm{vec}}\right) \mathrm{e}^{-\frac{1}{2}(\lambda_1 \cdot H u_1 + \lambda_2 \cdot H u_2)} \, \mathrm{e}^{\tilde{\psi}_1 E_{\alpha_1}} \end{aligned} \qquad (3.25)$$

where $g^f_{0,\mathrm{vec}} = \mathrm{e}^{-t_2 E_{-\alpha_2} - t_1 E_{-\alpha_1-\alpha_2}} \, \mathrm{e}^{\phi_1 h_1 + \phi_2 h_2} \, \mathrm{e}^{t_2 E_{\alpha_2} + t_1 E_{\alpha_1+\alpha_2}}$. We next choose u_1, u_2 such that

$$\begin{aligned} &\tilde{\chi}_2 \, \mathrm{e}^{-\frac{1}{2} u_2} = -t_2 & &\tilde{\psi}_2 \, \mathrm{e}^{\frac{1}{2} u_2} = t_2 \\ &\tilde{\chi}_3 \, \mathrm{e}^{-\frac{1}{2}(u_1+u_2)} = -t_1 & &\tilde{\psi}_3 \, \mathrm{e}^{\frac{1}{2}(u_1+u_2)} = t_1. \end{aligned}$$

Taking into account the parametrization (3.17) for $SL(3)$ we find

$$\begin{aligned} \mathrm{Tr}&\left(A_0^{(0)} \bar{A}_0^{(0)} - A_0^{(0)} g^f_{0,\mathrm{vec}} \bar{A}_0^{(0)} g^{f\,-1}_{0,\mathrm{vec}} + \bar{A}_0^{(0)} g^{f\,-1}_{0,\mathrm{vec}} \partial g^f_{0,\mathrm{vec}} - A_0^{(0)} \bar{\partial} g^f_{0,\mathrm{vec}} g^{f\,-1}_{0,\mathrm{vec}}\right) \\ &= a_1 \bar{a}_1 \bar{\Delta} + \bar{a}_1 (a_{01} t_1 t_2 + t_2 \partial t_1) \, \mathrm{e}^{\phi_1+\phi_2} + a_1 (\bar{a}_{01} t_1 t_2 - t_2 \bar{\partial} t_1) \, \mathrm{e}^{\phi_1+\phi_2} + a_{01} \bar{a}_{01} t_1^2 \, \mathrm{e}^{\phi_1+\phi_2} \\ &\quad + a_{02} \bar{a}_{02} t_2^2 \, \mathrm{e}^{2\phi_2-\phi_1} + \bar{a}_{01} (\partial\phi_1 + t_1 \partial t_1 \, \mathrm{e}^{\phi_1+\phi_2}) + \bar{a}_{02} (\partial\phi_2 - \partial\phi_1 + t_2 \partial t_2 \, \mathrm{e}^{-\phi_1+2\phi_2}) \\ &\quad - a_{01} (\bar{\partial}\phi_1 + t_1 \bar{\partial} t_1 \, \mathrm{e}^{\phi_1+\phi_2}) - a_{02} (\bar{\partial}\phi_2 - \bar{\partial}\phi_1 + t_2 \bar{\partial} t_2 \, \mathrm{e}^{-\phi_1+2\phi_2}) \end{aligned} \qquad (3.26)$$

where $\bar{\Delta} = t_2^2 \, \mathrm{e}^{\phi_1+\phi_2} - \mathrm{e}^{2\phi_1-\phi_2}$. We first take the integral over a_1 and $\bar{a}_1$ in the partition function (2.5) with the action given by (2.12). As a result we get

$$\mathcal{L}_{\mathrm{int}} = \bar{a}_{0i} M_{ij} a_{0j} + \bar{a}_{0i} N_i + \bar{N}_i a_{0i} + \frac{t_2^2 \partial t_1 \bar{\partial} t_1}{\bar{\Delta}} \, \mathrm{e}^{2(\phi_1+\phi_2)} \qquad (3.27)$$

where

$$M_{11} = -\frac{t_1^2}{\bar{\Delta}} \, \mathrm{e}^{3\phi_1} \qquad M_{22} = t_2^2 \, \mathrm{e}^{2\phi_2-\phi_1} \qquad M_{12} = M_{21} = 0 \qquad (3.28)$$

and

$$\begin{aligned} N_1 &= \partial\phi_1 + t_1 \partial t_1 \, \mathrm{e}^{\phi_1+\phi_2} - \frac{t_1 t_2^2 \partial t_1}{\bar{\Delta}} \, \mathrm{e}^{2(\phi_1+\phi_2)} & N_2 &= \partial\phi_2 - \partial\phi_1 + t_2 \partial t_2 \, \mathrm{e}^{-\phi_1+2\phi_2} \\ \bar{N}_1 &= -\bar{\partial}\phi_1 - t_1 \bar{\partial} t_1 \, \mathrm{e}^{\phi_1+\phi_2} + \frac{t_1 t_2^2 \bar{\partial} t_1}{\bar{\Delta}} \, \mathrm{e}^{2(\phi_1+\phi_2)} & \bar{N}_2 &= -\bar{\partial}\phi_2 + \bar{\partial}\phi_1 - t_2 \bar{\partial} t_2 \, \mathrm{e}^{-\phi_1+2\phi_2}. \end{aligned} \qquad (3.29)$$

We next integrate the fields $\bar{a}_{0i}$ and a_{0i}, $i = 1, 2$ in equation (3.27). Together with the standard form of WZW action $S_{\mathrm{WZW}}\left(g^f_{0,\mathrm{vec}}\right)$ we arrive at the following effective Lagrangian for the vector IM:

$$\begin{aligned} \mathcal{L}_{\mathrm{vec}} = \frac{1}{2} \sum_{i=1}^{2} \eta_{ij} \partial\phi_i \bar{\partial}\phi_j &+ \frac{\partial\phi_1 \bar{\partial}\phi_1}{t_1^2} \, \mathrm{e}^{-\phi_1-\phi_2} + \bar{\partial}\phi_1 \partial \ln(t_1) + \partial\phi_1 \bar{\partial} \ln(t_1) \\ &- \partial\phi_1 \bar{\partial}\phi_1 \left(\frac{t_2}{t_1}\right)^2 \mathrm{e}^{-2\phi_1+\phi_2} + \frac{\bar{\partial}(\phi_2 - \phi_1) \partial(\phi_2 - \phi_1)}{t_2^2} \, \mathrm{e}^{\phi_1-2\phi_2} \\ &+ \bar{\partial}(\phi_2 - \phi_1) \partial \ln(t_2) + \partial(\phi_2 - \phi_1) \bar{\partial} \ln(t_2) - V \end{aligned} \qquad (3.30)$$

where $V = \mu^2\left(\frac{2}{3} - t_2^2 \, \mathrm{e}^{-\phi_1+2\phi_2} - t_1^2 \, \mathrm{e}^{\phi_1+\phi_2}\right)$ and $\eta_{ij} = 2\delta_{ij} - \delta_{i,j-1} - \delta_{i,j+1}$. The integrability of the axial (3.24) and vector (3.30) models is a consequence of the Lax representation (2.15) and (2.16) valid for both models.

4. Local and global symmetries

Before imposing the subsidiary constraints (2.10), the model on the *group* G_0 described by (2.6) is invariant under *chiral* transformation (2.9) generated by $G_0^0 \otimes G_0^0$. For the explicit $SL(3)$ case, the associated Noether currents are given in terms of the axial variables defined in (3.21) as

$$\begin{aligned}
J_{-\alpha_1} &= \partial\tilde{\psi}_1 - \tilde{\psi}_1^2 \partial\tilde{\chi}_1\, \mathrm{e}^{R_1} + \partial\tilde{\chi}_2(\tilde{\psi}_1\tilde{\psi}_2 - \tilde{\psi}_3)\,\mathrm{e}^{R_2} \\
&\qquad + (\partial\tilde{\chi}_3 - \tilde{\chi}_2\partial\tilde{\chi}_1)(\tilde{\psi}_1\tilde{\psi}_2 - \tilde{\psi}_3)\tilde{\psi}_1\,\mathrm{e}^{R_1+R_2} + \tilde{\psi}_1\partial R_1 \\
J_{\alpha_1} &= \partial\tilde{\chi}_1\,\mathrm{e}^{R_1} - \tilde{\psi}_2(\partial\tilde{\chi}_3 - \tilde{\chi}_2\partial\tilde{\chi}_1)\,\mathrm{e}^{R_1+R_2} \\
J_{\lambda_1\cdot H} &= \tfrac{1}{3}(2\partial R_1 + \partial R_2) - \tilde{\psi}_1\partial\tilde{\chi}_1\,\mathrm{e}^{R_1} + (\tilde{\psi}_1\tilde{\psi}_2 - \tilde{\psi}_3)(\partial\tilde{\chi}_3 - \tilde{\chi}_2\partial\tilde{\chi}_1)\,\mathrm{e}^{R_1+R_2} \\
J_{\lambda_2\cdot H} &= \tfrac{1}{3}(\partial R_1 + 2\partial R_2) - \tilde{\psi}_2\partial\tilde{\chi}_2\,\mathrm{e}^{R_2} - \tilde{\psi}_3(\partial\tilde{\chi}_3 - \tilde{\chi}_2\partial\tilde{\chi}_1)\,\mathrm{e}^{R_1+R_2} \\
\bar{J}_{\alpha_1} &= \bar{\partial}\tilde{\chi}_1 - \tilde{\chi}_1^2\bar{\partial}\tilde{\psi}_1\,\mathrm{e}^{R_1} + \bar{\partial}\tilde{\psi}_2(\tilde{\chi}_1\tilde{\chi}_2 - \tilde{\chi}_3)\,\mathrm{e}^{R_2} \\
&\qquad + (\bar{\partial}\tilde{\psi}_3 - \tilde{\psi}_2\bar{\partial}\tilde{\psi}_1)(\tilde{\chi}_1\tilde{\chi}_2 - \tilde{\chi}_3)\tilde{\chi}_1\,\mathrm{e}^{R_1+R_2} + \tilde{\chi}_1\partial R_1 \\
\bar{J}_{-\alpha_1} &= \bar{\partial}\tilde{\psi}_1\,\mathrm{e}^{R_1} - \tilde{\chi}_2(\bar{\partial}\tilde{\psi}_3 - \tilde{\psi}_2\bar{\partial}\tilde{\psi}_1)\,\mathrm{e}^{R_1+R_2} \\
\bar{J}_{\lambda_1\cdot H} &= \tfrac{1}{3}(2\bar{\partial}R_1 + \bar{\partial}R_2) - \tilde{\chi}_1\bar{\partial}\tilde{\psi}_1\,\mathrm{e}^{R_1} + (\tilde{\chi}_1\tilde{\chi}_2 - \tilde{\chi}_3)(\bar{\partial}\tilde{\psi}_3 - \tilde{\psi}_2\bar{\partial}\tilde{\psi}_1)\,\mathrm{e}^{R_1+R_2} \\
\bar{J}_{\lambda_2\cdot H} &= \tfrac{1}{3}(\bar{\partial}R_1 + 2\bar{\partial}R_2) - \tilde{\chi}_2\bar{\partial}\tilde{\psi}_2\,\mathrm{e}^{R_2} - \tilde{\chi}_3(\bar{\partial}\tilde{\psi}_3 - \tilde{\psi}_2\bar{\partial}\tilde{\psi}_1)\,\mathrm{e}^{R_1+R_2}
\end{aligned} \tag{4.31}$$

where $\bar{\partial}J = \partial\bar{J} = 0$ and $J = J_{\lambda_1\cdot H}h_1 + J_{\lambda_2\cdot H}h_2 + \sum_\alpha J_\alpha E_{-\alpha} + J_{-\alpha}E_\alpha$, $\alpha = \alpha_1, \alpha_2, \alpha_1+\alpha_2$. Apart from those Noether currents (4.31) note the existence of *topological* currents

$$j_{\varphi,\mu} = \epsilon_{\mu\nu}\partial_\nu\varphi \qquad \varphi = \{R_i, i = 1, 2, \tilde{\chi}_j, \tilde{\psi}_j, j = 1, 2, 3\}. \tag{4.32}$$

The reduction from the group G_0 to the coset G_0/G_0^0 implies the vanishing of currents (4.31), which defines the unphysical non-local fields R_i in terms of ψ_i, χ_i:

$$\begin{aligned}
\partial R_1 &= \frac{\psi_1\partial\chi_1}{\Delta}\left(1 + \frac{3}{2}\psi_2\chi_2\right) - \frac{\psi_2\partial\chi_2}{\Delta}\left(\Delta_2 + \frac{3}{2}\psi_1\chi_1\right) \\
\partial R_2 &= \frac{\psi_1\partial\chi_1}{\Delta} + \frac{\psi_2\partial\chi_2}{\Delta}\left(2\Delta_2 + \frac{3}{2}\psi_1\chi_1\right) \\
\bar{\partial} R_1 &= \frac{\chi_1\bar{\partial}\psi_1}{\Delta}\left(1 + \frac{3}{2}\psi_2\chi_2\right) - \frac{\chi_2\bar{\partial}\psi_2}{\Delta}\left(\Delta_2 + \frac{3}{2}\psi_1\chi_1\right) \\
\bar{\partial} R_2 &= \frac{\chi_1\bar{\partial}\psi_1}{\Delta} + \frac{\chi_2\bar{\partial}\psi_2}{\Delta}\left(2\Delta_2 + \frac{3}{2}\psi_1\chi_1\right)
\end{aligned} \tag{4.33}$$

where $\Delta = (1+\psi_2\chi_2)^2 + \psi_1\chi_1\left(1 + \frac{3}{4}\psi_2\chi_2\right)$, $\Delta_2 = 1 + \psi_2\chi_2$ and

$$\begin{aligned}
&\tilde{\chi}_1 = \chi_3\,\mathrm{e}^{-\frac{1}{2}R_1} \qquad \tilde{\psi}_1 = \psi_3\,\mathrm{e}^{-\frac{1}{2}R_1} \qquad \tilde{\chi}_2 = \chi_2\,\mathrm{e}^{-\frac{1}{2}R_2} \qquad \tilde{\psi}_2 = \psi_2\,\mathrm{e}^{-\frac{1}{2}R_2} \\
&\tilde{\chi}_3 = \chi_1\,\mathrm{e}^{-\frac{1}{2}(R_1+R_2)} \qquad \tilde{\psi}_3 = \psi_1\,\mathrm{e}^{-\frac{1}{2}(R_1+R_2)}.
\end{aligned} \tag{4.34}$$

In addition we find

$$\begin{aligned}
\partial\tilde{\chi}_1 &= \frac{\psi_2}{\Delta}\left(\partial\chi_1\Delta_2 - \frac{1}{2}\chi_1\psi_2\partial\chi_2\right)\mathrm{e}^{-\frac{1}{2}R_1} \\
\partial\tilde{\psi}_1 &= \frac{\psi_1}{\Delta}\left(\partial\chi_2(1 + \psi_1\chi_1 + \psi_2\chi_2) - \frac{1}{2}\chi_2\psi_1\partial\chi_1\right)\mathrm{e}^{-\frac{1}{2}R_1} \\
\bar{\partial}\tilde{\psi}_1 &= \frac{\chi_2}{\Delta}\left(\bar{\partial}\psi_1\Delta_2 - \frac{1}{2}\psi_1\chi_2\bar{\partial}\psi_2\right)\mathrm{e}^{-\frac{1}{2}R_1} \\
\bar{\partial}\tilde{\chi}_1 &= \frac{\chi_1}{\Delta}\left(\bar{\partial}\psi_2(1 + \psi_1\chi_1 + \psi_2\chi_2) - \frac{1}{2}\chi_1\psi_2\bar{\partial}\psi_1\right)\mathrm{e}^{-\frac{1}{2}R_1}.
\end{aligned} \tag{4.35}$$

Using the equations of motion derived from (3.24), we prove the following conservation laws[3]:

$$\bar{\partial} j = \partial \bar{j} \qquad j = j_{\tilde{\psi}_1}, j_{\tilde{\chi}_1} \qquad j = j_{R_i} \quad i = 1, 2 \tag{4.36}$$

where $j = \frac{1}{2}(j_0 + j_1)$, $\bar{j} = \frac{1}{2}(j_0 - j_1)$ and

$$j_{R_i,\mu} = \epsilon_{\mu\nu}\partial_\nu R_i \quad i = 1, 2 \qquad j_{\tilde{\psi}_1,\mu} = \epsilon_{\mu\nu}\partial_\nu \tilde{\psi}_1 \qquad j_{\tilde{\chi}_1,\mu} = \epsilon_{\mu\nu}\partial_\nu \tilde{\chi}_1. \tag{4.37}$$

Under the reduction (2.10), the topological currents (4.32) in the group G_0 become Noether currents (4.37) in the coset G_0/G_0^0 and their conservation is a consequence of the invariance of action (3.24) under the following non-local global transformations:

$$\begin{aligned}
\delta\psi_1 &= \tfrac{1}{2}(-\epsilon_1 - \epsilon_2 + \bar{\epsilon}_1 + \bar{\epsilon}_2)\psi_1 - \tfrac{1}{2}\epsilon_-\psi_1\tilde{\psi}_1 + \bar{\epsilon}_+\left(\psi_2\,\mathrm{e}^{-\frac{1}{2}R_1} + \tfrac{1}{2}\psi_1\tilde{\chi}_1\right)\\
\delta\chi_1 &= \tfrac{1}{2}(\epsilon_1 + \epsilon_2 - \bar{\epsilon}_1 - \bar{\epsilon}_2)\chi_1 - \tfrac{1}{2}\bar{\epsilon}_+\chi_1\tilde{\chi}_1 + \epsilon_-\left(\chi_2\,\mathrm{e}^{-\frac{1}{2}R_1} + \tfrac{1}{2}\chi_1\tilde{\psi}_1\right)\\
\delta\psi_2 &= \epsilon_-\left(\tfrac{1}{2}\psi_2\tilde{\psi}_1 - \psi_1\,\mathrm{e}^{-\frac{1}{2}R_1}\right) - \tfrac{1}{2}\bar{\epsilon}_+\psi_2\tilde{\chi}_1 + \tfrac{1}{2}(-\epsilon_2 + \bar{\epsilon}_2)\psi_2\\
\delta\chi_2 &= \bar{\epsilon}_+\left(\tfrac{1}{2}\chi_2\tilde{\chi}_1 - \chi_1\,\mathrm{e}^{-\frac{1}{2}R_1}\right) - \tfrac{1}{2}\epsilon_-\chi_2\tilde{\psi}_1 + \tfrac{1}{2}(\epsilon_2 - \bar{\epsilon}_2)\chi_2
\end{aligned} \tag{4.38}$$

where $\epsilon_1 - \bar{\epsilon}_1, \epsilon_2 - \bar{\epsilon}_2, \epsilon_-$ and $\bar{\epsilon}_+$ are arbitrary constants. The algebra of such transformations can be shown to be the q-deformed Poisson bracket algebra $SL(2)_q \otimes U(1)$ [19], with $q = \exp\left(-\frac{2\pi}{k}\right)$. The global symmetries of the vector model generate the same algebra.

5. Non-conformal T-duality

T-duality in the context of the conformal σ-models

$$S_\sigma^{\mathrm{conf}} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2 z\left((g_{MN}(X)\eta^{\mu\nu} + \epsilon^{\mu\nu}b_{MN}(X))\partial_\mu X^M\partial_\nu X^N + \frac{\alpha'}{2}R^{(2)}\varphi(X)\right) \tag{5.39}$$

($\mu, \nu = 0, 1, M, N = 1, 2, \ldots D$ and $R^{(2)}$ is the worldsheet curvature), represents specific canonical transformations (CT): $\left(\Pi_{X_M}, X^M\right) \to \left(\Pi_{\tilde{X}_M}, \tilde{X}^M\right)$ that map (5.39) into its dual σ-model $S_\sigma^{\mathrm{conf}}(G_{M,N}(\tilde{X}), B_{M,N}(\tilde{X}), \phi(\tilde{X}))$. In the case of curved backgrounds with d-isometric directions (i.e. the metric $g_{MN}(X^m)$, the antisymmetric tensor $b_{MN}(X^m)$ and the dilaton $\varphi(X^m)$ are independent of the $d \leqslant D$ fields $X_\alpha(z, \bar{z})$, $\alpha = 1, 2, \ldots d$) the corresponding CT has the form:

$$\Pi_{\tilde{X}_\alpha} = -2\partial_x X_\alpha \qquad \Pi_{X_\alpha} = -2\partial_x \tilde{X}_\alpha \tag{5.40}$$

and the other Π_{X_m} and $X_m, m = d+1, \ldots D$ remain unchanged. Then T-duality manifests as (matrix) transformations of the target-space geometry data of (5.39): $e_{MN}(X) = b_{MN}(X) + g_{MN}(X)$ and $\varphi(X)$ to its T-dual $E_{MN}(\tilde{X}) = B_{MN}(\tilde{X}) + G_{MN}(\tilde{X})$ and $\phi(\tilde{X})$ [20]:

$$\begin{aligned}
&E_{\alpha\beta} = (\mathrm{e}^{-1})_{\alpha\beta} && E_{mn} = e_{mn} - e_{m\alpha}(\mathrm{e}^{-1})^{\alpha\beta}e_{\beta n} &&\\
&E_{\alpha m} = (\mathrm{e}^{-1})_\alpha^\beta e_{\beta m} && E_{m\alpha} = -e_{m\beta}(\mathrm{e}^{-1})_\alpha^\beta && \phi = \varphi - \ln(\det e_{\alpha\beta}).
\end{aligned} \tag{5.41}$$

By construction the dual pair of σ-models $S_\sigma^{\mathrm{conf}}(e, \varphi)$ and $\tilde{S}_\sigma^{\mathrm{conf}}(E, \phi)$ share the same spectra and partition functions. Their Lagrangians are related by the generating function $\mathcal{F}$ [1]

$$\mathcal{L}(e, \varphi) = \mathcal{L}(E, \phi) + \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} \qquad \mathcal{F} = \frac{1}{8\pi\alpha'}\int \mathrm{d}x(X\cdot\partial_x\tilde{X} - \partial_x X\cdot\tilde{X}). \tag{5.42}$$

An important feature of the Abelian T-duality (5.40) and (5.41) is that it maps the $U(1)^{\otimes d}$ Noether charges $Q^\alpha = \int_{-\infty}^{\infty} J_o^\alpha\,\mathrm{d}x$ of $S_\sigma^{\mathrm{conf}}(e, \varphi)$ into the topological charges

[3] Note that (4.35) denotes non-local fields $R_1, R_2, \tilde{\psi}_1, \tilde{\chi}_1$ in terms of the physical fields ψ_1, ψ_2, χ_1 and χ_2 and hence conservation of (4.37) is non-trivial.

$\tilde{Q}^{\alpha}_{\text{top}} = \int_{-\infty}^{\infty} \partial_x \tilde{X}^{\alpha}\, \mathrm{d}x$ of its T-dual model $\tilde{S}^{\text{conf}}_{\sigma}(E, \phi)$ and vice versa, i.e. we have

$$\begin{aligned} J^{\alpha}_{\mu} &= e^{\alpha\beta}(X_n)\partial_{\mu}X_{\beta} + e^{\alpha m}(X_n)\partial_{\mu}X_m = \epsilon_{\mu\nu}\partial^{\nu}\tilde{X}^{\alpha} \\ \tilde{J}^{\alpha}_{\mu} &= E^{\alpha\beta}(\tilde{X}_n)\partial_{\mu}\tilde{X}_{\beta} + E^{\alpha m}(\tilde{X}_n)\partial_{\mu}\tilde{X}_m = \epsilon_{\mu\nu}\partial^{\nu}X^{\alpha} \end{aligned} \tag{5.43}$$

and therefore

$$T : \left(Q^{\alpha}, Q^{\alpha}_{\text{top}}\right) \to \left(\tilde{Q}^{\alpha}_{\text{top}}, \tilde{Q}^{\alpha}\right).$$

Different examples of such T-dual pairs of conformal σ-models have been constructed in terms of axial and vector gauged G/H-WZW models (see [4] and references therein).

On the other hand, the IMs considered in sections 2 and 3 have as their conformal limits ($\mu = 0$, i.e. $V = 0$ in (3.24) and (3.30)) the corresponding axial and vector gauged $SL(3, R)/SL(2, R)\otimes U(1)$-WZW models which are T-dual by construction. They have $d = 2$ isometric directions, i.e. $e_{MN}(\psi_i, \chi_i)$ are independent of $\Theta_i = \ln\left(\frac{\psi_i}{\chi_i}\right)$. The T-duality group in this case is known to be $O(2, 2|Z)$ (see for instance [2]). The problem we address in this section is about T-duality of the IMs (3.24) and (3.30). We first note the important property of these IMs, namely adding the potentials $V = \text{Tr}\left(\epsilon_+ g_0^f \epsilon_- \left(g_0^f\right)^{-1}\right)$ breaks the conformal symmetry, but one still keeps two isometries, i.e., $U(1) \otimes U(1)$ invariance, say $\Theta_i \to \Theta_i + \alpha_i$ in the axial case. This suggests that the T-duality of the conformal G/H-WZW models can be extended to T-duality for their integrable perturbations (3.24) and (3.30). In order to prove it we extend the Buscher procedure [20] of deriving the T-dual of a given conformal σ-model (with d isometries) to the case of IMs, i.e. in the presence of the potential $V(X_n)$.

5.1. Isometries and T-dual actions

Let us consider the Lagrangian density of the form

$$\mathcal{L}^{\text{ax}}_{\text{IM}} = \mathcal{L}^{\text{conf}}_{\sigma}(\Theta_{\alpha}, X_m) - V(X_m) \tag{5.44}$$

where $\mathcal{L}^{\text{conf}}_{\sigma}$ is the Lagrangian (5.39) with $X_{\alpha} = \Theta_{\alpha}$ and the potential $V(X_m)$ is independent of Θ_{α}. We next rewrite (5.39) in a symbolic form separating the isometric fields $\Theta_{\alpha}, \alpha = 1, 2, \ldots d$ from the remaining ones $X_m, m = d + 1, \ldots D$:

$$\mathcal{L}^{\text{ax}}_{\text{IM}} = \bar{\partial}\Theta_{\alpha}e^{\alpha\beta}(X_m)\partial\Theta_{\beta} + \bar{\partial}\Theta_{\alpha}N_{\alpha} + \bar{N}_{\alpha}\partial\Theta_{\alpha} + \mathcal{L}'(X_m). \tag{5.45}$$

In order to derive $\mathcal{L}^{\text{vec}}_{\text{IM}}(\tilde{\Theta}_{\alpha}, \tilde{X}_m)$ of the T-dual IM we apply equation (5.42), i.e.

$$\mathcal{L}^{\text{vec}}_{\text{IM}}(\tilde{\Theta}_{\alpha}, \tilde{X}_m) = \mathcal{L}^{\text{ax}}_{\text{IM}}(\Theta_{\alpha}, X_m) - \tilde{\Theta}_{\alpha}(\partial\bar{P}_{\alpha} - \bar{\partial}P_{\alpha}) \tag{5.46}$$

where we denote $P_{\alpha} = \partial\Theta_{\alpha}$, $\bar{P}_{\alpha} = \bar{\partial}\Theta_{\alpha}$ and the second term is nothing but the contribution of the generating function $\mathcal{F}(\Theta_{\alpha}, \tilde{\Theta}_{\alpha}) \sim \epsilon^{\mu\nu}\partial_{\mu}\Theta_{\alpha}\partial_{\nu}\tilde{\Theta}^{\alpha}$. We first integrate (5.46) by parts

$$\mathcal{L}^{\text{vec}}_{\text{IM}} = \bar{P}_{\alpha}e_{\alpha\beta}P_{\beta} + \bar{P}_{\alpha}(N_{\alpha} + \partial\tilde{\Theta}_{\alpha}) + (\bar{N}_{\alpha} - \bar{\partial}\tilde{\Theta}_{\alpha})P_{\alpha} + \mathcal{L}'(X_m) \tag{5.47}$$

and next we can take the Gaussian integral in $\bar{P}_a$ and P_{α} in the corresponding path integral. Therefore, the effective action for the T-dual model has the form

$$\mathcal{L}^{\text{vec}}_{\text{IM}}(\tilde{\Theta}_{\alpha}, X_m) = -(\bar{N}_{\alpha} - \bar{\partial}\tilde{\Theta}_{\alpha})\, e^{-1}_{\alpha\beta}(N_{\beta} + \partial\tilde{\Theta}_{\beta}) + \mathcal{L}'(X_m) - 4\pi(\alpha')^2 \ln(\det e_{\alpha\beta})R^{(2)} \tag{5.48}$$

in accordance with equations (5.41).

The second question to be addressed is whether the Lagrangians (5.44) and (5.48) are related by canonical transformations (5.40). In order to answer it, we shall compare their Hamiltonians:

$$\mathcal{H}^{\text{ax}} = \dot{\Theta}_{\alpha}\Pi_{\Theta_{\alpha}} + \dot{X}_m\Pi_{X_m} - \mathcal{L}^{\text{ax}} \qquad \mathcal{H}^{\text{vec}} = \dot{\tilde{\Theta}}_{\alpha}\Pi_{\tilde{\Theta}_{\alpha}} + \dot{X}_m\Pi_{X_m} - \mathcal{L}^{\text{vec}}$$

since by definition

$$\Pi_{\Theta_\alpha} = \frac{\delta \mathcal{L}^{\mathrm{ax}}}{\delta \dot{\Theta}_\alpha} = 2\dot{\Theta}_\beta e_{\alpha\beta} + N_\alpha + \bar{N}_\alpha \qquad \Pi_{\tilde{\Theta}_\alpha} = \frac{\delta \mathcal{L}^{\mathrm{vec}}}{\delta \dot{\tilde{\Theta}}_\alpha} = e^{-1}_{\alpha\beta}(2\dot{\tilde{\Theta}}_\beta + N_\beta - \bar{N}_\beta) \tag{5.49}$$

we find that

$$\mathcal{H}^{\mathrm{ax}} = \tfrac{1}{4}\Pi_{\Theta_\alpha} e^{-1}_{\alpha\beta}\Pi_{\Theta_\beta} - \tfrac{1}{2}\Pi_{\Theta_\alpha} e^{-1}_{\alpha\beta}(N_\beta + \bar{N}_\beta) + \partial_x \Theta_\alpha e_{\alpha\beta}\partial_x \Theta_\beta + \partial_x \Theta_\alpha (N_\alpha - \bar{N}_\alpha) \\ + \tfrac{1}{4}(N_i + \bar{N}_i)e^{-1}_{ij}(N_j + \bar{N}_j) + \mathcal{H}\big(X_m, \Pi_{X_m}\big) \tag{5.50}$$

and

$$\mathcal{H}^{\mathrm{vec}} = \tfrac{1}{4}\Pi_{\tilde{\Theta}_\alpha} e_{\alpha\beta}\Pi_{\tilde{\Theta}_\beta} - \tfrac{1}{2}\Pi_{\tilde{\Theta}_\alpha}(N_\alpha - \bar{N}_\alpha) + \partial_x \tilde{\Theta}_\alpha e^{-1}_{\alpha\beta}\partial_x \tilde{\Theta}_\beta + \partial_x \tilde{\Theta}_\alpha e^{-1}_{\alpha\beta}(N_\beta + \bar{N}_\beta) \\ + \tfrac{1}{4}(N_i + \bar{N}_i)e^{-1}_{ij}(N_j + \bar{N}_j) + \mathcal{H}\big(X_m, \Pi_{X_m}\big) \tag{5.51}$$

where $\mathcal{H}\big(X_m, \Pi_{X_m}\big) = \dot{X}_m \Pi_{X_m} - \mathcal{L}'(X_m)$. Finally we observe that $\mathcal{H}^{\mathrm{ax}} = \mathcal{H}^{\mathrm{vec}}$, i.e. integrable models (5.44) and (5.48) have coinciding Hamiltonians if the transformation

$$\Pi_{\Theta_\alpha} = -2\partial_x \tilde{\Theta}_\alpha \qquad \Pi_{\tilde{\Theta}_\alpha} = -2\partial_x \Theta_\alpha \tag{5.52}$$

takes place. This is precisely the canonical transformation (5.40) relating the T-dual pairs of σ-models.

5.2. Axial–vector duality for homogeneous grading models

In order to prove that the axial (3.24) and vector (3.30) IMs are T-dual to each other, we apply the procedure explained in section 5.1. Starting from equation (3.24) we recognize the two isometric 'coordinates' to be $\Theta_\alpha = \ln\big(\frac{\psi_\alpha}{\chi_\alpha}\big)$, $\alpha = 1, 2$. By changing variables

$$\psi_\alpha, \chi_\alpha \to \Theta_\alpha \qquad a_m = \psi_m \chi_m \quad m = 1, 2$$

one can rewrite $\mathcal{L}^{\mathrm{ax}}$ in (3.24) in the form (5.45) with

$$\mathcal{L}'(X_m) = \frac{\bar{\partial} a_1 \partial a_1}{4\Delta a_1}(1 + a_2) + \frac{\bar{\partial} a_2 \partial a_2}{4\Delta a_2}(1 + a_1 + a_2) - \frac{\bar{\partial} a_1 \partial a_2}{8\Delta} - \frac{\bar{\partial} a_2 \partial a_1}{8\Delta} - \mu^2\left(\frac{2}{3} + a_1 + a_2\right)$$

and

$$e_{11} = -\frac{1}{4\Delta}(1 + a_2)a_1 \qquad e_{22} = -\frac{1}{4\Delta}(1 + a_1 + a_2)a_2 \qquad e_{12} = e_{21} = \frac{1}{8\Delta}a_1 a_2$$

$$N_1 = \frac{1}{4\Delta}\left((1 + a_2)\partial a_1 - \frac{1}{2}a_1 \partial a_2\right) \qquad N_2 = \frac{1}{4\Delta}\left((1 + a_1 + a_2)\partial a_2 - \frac{1}{2}a_2 \partial a_1\right) \tag{5.53}$$

$$\bar{N}_1 = \frac{1}{4\Delta}\left(-(1 + a_2)\bar{\partial} a_1 + \frac{1}{2}a_1 \bar{\partial} a_2\right) \qquad \bar{N}_2 = -\frac{1}{4\Delta}\left((1 + a_1 + a_2)\bar{\partial} a_2 - \frac{1}{2}a_2 \bar{\partial} a_1\right).$$

Therefore, according to equations (5.46) and (5.47) the axial and vector IMs are related by canonical transformation (5.52). The identification of $\mathcal{L}^{\mathrm{vec}}_{\mathrm{IM}}$ in (5.48) with the vector model Lagrangian (3.30) becomes evident by observing the relations among the fields,

$$a_1 = -t_1^2\,\mathrm{e}^{\phi_1 + \phi_2} \qquad a_2 = -t_2^2\,\mathrm{e}^{-\phi_1 + 2\phi_2} \qquad \tilde{\Theta}_1 = -\tfrac{1}{2}\phi_1 \qquad \tilde{\Theta}_2 = -\tfrac{1}{2}(\phi_2 - \phi_1). \tag{5.54}$$

Another important feature of the axial–vector T-duality is the simple relation between the isometric fields $\tilde{\Theta}_\alpha$ of the vector model (3.30) and the non-local fields R_i (see (4.33)) of the axial model,

$$R_1 = 2(\tilde{\Theta}_2 - \tilde{\Theta}_1) \qquad R_2 = -2(\tilde{\Theta}_1 + 2\tilde{\Theta}_2). \tag{5.55}$$

The above identification can be established by solving the constraints (2.10) (or in the explicit form (4.33) for the $SL(3)$ case) in favour of the non-local fields of the vector model Θ_i:

$$\begin{aligned}
\partial\Theta_1 &= \partial \ln a_1 - \partial(R_1 + R_2) - \frac{2}{3}\frac{a_2+1}{a_1}\partial(2R_1 + R_2) \\
\partial\Theta_2 &= \partial \ln a_2 + \frac{2}{3}\frac{a_2+1}{a_2}\partial(R_1 - R_2) - \frac{1}{3}\partial(2R_1 + R_2) \\
\bar\partial\Theta_1 &= -\bar\partial \ln a_1 + \bar\partial(R_1 + R_2) + \frac{2}{3}\frac{a_2+1}{a_1}\bar\partial(2R_1 + R_2) \\
\bar\partial\Theta_2 &= -\bar\partial \ln a_2 - \frac{2}{3}\frac{a_2+1}{a_2}\bar\partial(R_1 - R_2) + \frac{1}{3}\bar\partial(2R_1 + R_2)
\end{aligned} \tag{5.56}$$

and next comparing the RHS of equation (5.56) with the $U(1)\otimes U(1)$ conserved currents of the vector model Lagrangian (3.30). We can further write equations (5.56) and (4.33) in the compact form

$$J^{i,\mathrm{ax}}_{\mathrm{top}} = \epsilon_{\mu\nu}\partial^\nu\Theta_i = \tilde J^{i,\mathrm{vec}}_\mu \qquad \tilde J^{i,\mathrm{vec}}_{\mathrm{top}} = \epsilon_{\mu\nu}\partial^\nu R_i = J^{i,\mathrm{ax}}_\mu \tag{5.57}$$

or equivalently

$$\tilde I^{1,\mathrm{vec}}_{\mathrm{top}} = \epsilon_{\mu\nu}\partial^\nu\tilde\Theta_1 = -\tfrac{1}{6}\big(J^{2,\mathrm{ax}}_\mu + 2J^{1,\mathrm{ax}}_\mu\big) \qquad \tilde I^{2,\mathrm{vec}}_{\mathrm{top}} = \epsilon_{\mu\nu}\partial^\nu\tilde\Theta_2 = \tfrac{1}{6}\big(J^{1,\mathrm{ax}}_\mu - J^{2,\mathrm{ax}}_\mu\big). \tag{5.58}$$

For the $SL(3)$-case in consideration these equations exemplify the main property (5.43) of the T-dual pairs of models

$$Q^{\alpha,\mathrm{ax}}_{\mathrm{top}} = Q^{\alpha,\mathrm{vec}} \qquad Q^{\alpha,\mathrm{vec}}_{\mathrm{top}} = Q^{\alpha,\mathrm{ax}} \tag{5.59}$$

namely that T-duality relates the topological charges $Q^{\alpha,\mathrm{vec}}_{\mathrm{top}} = \int \mathrm{d}x\,\partial_x\tilde\Theta_\alpha$ to the $U(1)\otimes U(1)$-charges $Q^{\alpha,\mathrm{ax}}$ of the axial IM and vice versa.

An explicit realization of the above exchange of topological and $U(1)$-Noether charges (similar to the momentum-winding numbers exchange in string theory) has been observed in [7], analysing the 1-soliton structure spectrum of the corresponding dyonic IM. The masses of the solitons of axial and vector models remain equal, but the $U(1)$ charge of the axial non-topological solitons is transformed into the topological charge of the vector model solitons. Similar relations take place in the pair of T-dual non-Abelian dyonic models (3.24) and (3.30) in consideration [19].

6. Conclusions

We have demonstrated how one can extend the Abelian T-duality of the conformal gauged G/H-WZW models to their integrable perturbations, which appears to be identical to specific homogeneous gradation NA affine Toda models. More general considerations (presented in section 5) of generic (relativistic) IMs (as well as for non-integrable models) admitting isometric directions (i.e. with few global $U(1)$ symmetries) make it evident that one can construct their T-dual partners by appropriately chosen canonical transformations. The most important new feature of the T-duality in the context of 2D integrable models consists in its action on the spectrum of the solitons of the corresponding pair of dual IMs. As one can expect it maps the $U(1)^{\otimes d}$-charges of the solitons of the axial model (with d-isometries) to the topological charges of the solitons of its T-dual counterpart, leaving the soliton masses unchanged.

The quantization of the NA affine Toda models usually requires non-trivial counterterms [7, 10, 21] together with the renormalization of the couplings and masses. Hence, an interesting open problem is whether the quantum vector and axial IMs continue to be T-dual to each other.

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